New identities for the partial Bell polynomials

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Abstract. A new explicit closed-form formula for the multivariate (n, k)th partial Bell polynomial $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ is deduced. The formula involves multiple summations and makes it possible, for the first time, to easily evaluate $B_{n,k}$ directly for given values of n and k ($n \geq k, k = 2, 3, \ldots$). Also, a new addition formula (with respect to k) is found for the polynomials $B_{n,k}$ and it is shown that they admit a new recurrence relation. Several special cases and consequences are pointed out, and some examples are also given.

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1. Introduction

For n and k non-negative integers, the (exponential) (n, k)th partial Bell polynomial in the variables $x_1, x_2, \ldots, x_{n-k+1}$ denoted by $B_{n,k} \equiv B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ may be defined by the formal power series expansion (see, for instance, [1, pp. 133, Eq. (3a')])

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \frac{t^n}{n!} \qquad (k \geq 0), \tag{1.1}$$

or, what amounts to the same, by the explicit formula [2, p. 96]

$$B_{n,k} = \sum \frac{n!}{\ell_1!\,\ell_2!\ldots\ell_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{\ell_1}\left(\frac{x_2}{2!}\right)^{\ell_2}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{\ell_{n-k+1}}, \tag{1.2}$$

where (multiple) summation is extended over all partitions of a positive integer number n into exactly k parts (summands), *i.e.*, over all solutions in non-negative integers ℓ_α, $1 \leq \alpha \leq n-k+1$, of a system of the two simultaneous equations

$$\ell_1 + 2\,\ell_2 + \cdots + (n-k+1)\,\ell_{n-k+1} = n$$

and

$$\ell_1 + \ell_2 + \cdots + \ell_{n-k+1} = k.$$

For fixed n and k, $B_{n,k}$ has positive integral coefficients and is a homogenous and isobaric polynomial in its $(n-k+1)$ variables $x_1, x_2, \ldots, x_{n-k+1}$ of total degree k and total weight n, *i.e.*, it is a linear combination of monomials $x_1^{\ell_1} x_2^{\ell_2} \ldots x_{n-k+1}^{\ell_{n-k+1}}$ whose partial degrees and weights are constantly given by $\ell_1 + \ell_2 + \ldots + \ell_{n-k+1} = k$ and $\ell_1 + 2\ell_2 + \ldots + (n-k+1)\ell_{n-k+1} = n$. For some examples of these polynomials see Section 3.

The partial Bell polynomials are quite general polynomials, they have a number of applications and more details about them can be found in Bell [3], Comtet [1, pp. 133–137], Hazewinkel [2, pp. 95–98], Charalambides [4, pp. 412–417] and Aldrovandi [5, pp. 151–182]. However, the following formulae for $B_{n,k}$

$$B_{n,k} = \frac{1}{x_1} \cdot \frac{1}{n-k} \sum_{\alpha=1}^{n-k} \binom{n}{\alpha} \left[(k+1) - \frac{n+1}{\alpha+1} \right] x_{\alpha+1} B_{n-\alpha,k}, \tag{1.3}$$

$$B_{n,k_1+k_2} = \frac{k_1!\, k_2!}{(k_1+k_2)!} \sum_{\alpha=0}^{n} \binom{n}{\alpha} B_{\alpha,k_1} B_{n-\alpha,k_2} \tag{1.4}$$

and

$$B_{n,k+1} = \frac{1}{(k+1)!} \underbrace{\sum_{\alpha_1=k}^{n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k} \overbrace{\binom{n}{\alpha_1}\binom{\alpha_1}{\alpha_2} \cdots \binom{\alpha_{k-1}}{\alpha_k}}^{k}$$

$$\cdot\, x_{n-\alpha_1} x_{\alpha_1-\alpha_2} \cdots x_{\alpha_{k-1}-\alpha_k} x_{\alpha_k} \qquad (n \geq k+1, k = 1, 2, \ldots) \tag{1.5}$$

appear not to have been noticed in any work on the subject which we have seen. In this note it is aimed to provide short proofs of these results, show some immediate consequences of them and provide some application examples (see also Section 3).

2. Proof of the main results

We begin by showing that the identity (1.3) follows without difficulty from the definition of partial Bell polynomials $B_{n,k}$ by means of the generating relation (1.1), given that the next auxiliary result for powers of series is used. Consider

$$\left(\sum_{n=1}^{\infty} f_n x^n \right)^k = \sum_{n=k}^{\infty} g_n(k)\, x^n. \tag{2.1}$$

For a fixed positive integer k, we have that:

$$g_k(k) = f_1^k, \tag{2.2a}$$

$$g_n(k) = \frac{1}{(n-k) f_1} \sum_{\alpha=1}^{n-k} \Big[(\alpha+1)(k+1) - (n+1) \Big] f_{\alpha+1}\, g_{n-\alpha}(k) \tag{2.2b}$$

$$(n \geq k+1).$$

Indeed, by comparing (2.1) with the definition of $B_{n,k}$ in (1.1) and upon setting $g_n(k) = k! B_{n,k}/n!$ and $f_n = x_n/n!$, we arrive at the proposed formula (1.3) by utilizing (2.2b).

Note that (2.2b) may be found in the literature (see [6]) but it is not as widely known (and even less used) as it should be. It is exactly for this reason that we derive it starting from the following more general (and equally little known) recurrence relation involving the series coefficients f_n and $g_n(k)$ in $(\sum_{n=0}^{\infty} f_n x^n)^k = \sum_{n=0}^{\infty} g_n(k)\, x^n$.

$$\sum_{\alpha=0}^{n} \big[\alpha(k+1) - n \big] f_\alpha\, g_{n-\alpha}(k) = 0 \qquad (n \geq 0). \tag{2.3}$$

First, upon taking logarithms of each side of the equation $g(x) = \big[f(x)\big]^k$ and then differentiating both sides of the result with respect to x, we obtain $f(x)g'(x) = k\, f'(x)g(x)$. Next, insert the power series expansions of the various functions in this equation and multiply both sides by x, to get

$$\sum_{m=0}^{\infty} f_m x^m \cdot \sum_{m=0}^{\infty} m\, g_m(k)\, x^m = k \sum_{m=0}^{\infty} m\, f_m x^m \cdot \sum_{m=0}^{\infty} g_m(k)\, x^m. \tag{2.4}$$

Now, recall that if $\sum_{m=0}^{\infty} a_m$ and $\sum_{m=0}^{\infty} b_m$ are two series, then their Cauchy product is the series $\sum_{n=0}^{\infty} c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. This is to say that in the particular case at hand, by equating the coefficients of a given power of x, say x^n, on both sides of (2.4), we have $\sum_{\alpha=0}^{n} (n-\alpha) f_\alpha g_{n-\alpha}(k) = k \sum_{\alpha=0}^{n} \alpha f_\alpha g_{n-\alpha}(k)$, which eventually gives (2.3). The recurrence relation (2.3) is clearly valid for an arbitrary real or complex number k and it can be used to compute successively as many of the unknown $g_m(k)$ values as desired, in order $g_0(k), g_1(k), g_2(k), \ldots$, if $g_0(k)$ is known. The special case of (2.3) solved for $g_n(k)$, for k a positive integer and $f_0 \neq 0$, appears in various editions of the standard reference book by Gradshteyn and Ryzhik (see, for instance, [7, p. 17, Entry 0.314])

Finally, if we suppose $f_0 = 0$ and $f_1 \neq 0$ then, from $\big(\sum_{n=0}^{\infty} f_n x^n\big)^k = \sum_{n=0}^{\infty} g_n(k)\, x^n$, where k is a positive integer, it is obvious that the coefficient $g_n(k)$, $n = 0, 1, \ldots, k$, is only nonzero when $n = k$, $g_k(k)$ then equals f_1^k (see (2.2a)), while $\big(\sum_{n=0}^{\infty} f_n x^n\big)^k = \sum_{n=0}^{\infty} g_n(k)\, x^n$ reduces to (2.1). Therefore, since $g_0(k) = g_1(k) = \ldots = g_{k-1}(k) = 0$ and $f_0 = 0$, the recurrence relation (2.3) becomes

$$\sum_{\alpha=1}^{n-k} \big[\alpha(k+1) - n\big] f_\alpha\, g_{n-\alpha}(k) = 0 \qquad (n \geq k),$$

so that, upon replacing n by $n+1$, putting $\alpha+1$ for α and solving for $g_n(k)$, we have that the coefficients $g_n(k)$, $n \geq k+1$, are given by (2.2b) above.

In order to prove (1.4) we shall again resort to the generating relation for $B_{n,k}$ (1.1). Let us by $[t^n]\phi(t)$ denote the coefficient of t^n in the power series of an arbitrary $\phi(t)$. Put $f(t) = \sum_{m=1}^{\infty} x_m \frac{t^m}{m!}$, then by (1.1), we have

$$k_1!B_{n,k_1} = n!\,[t^n] f(t)^{k_1} \qquad (n \geq k_1)$$

and

$$k_2!B_{n,k_2} = n!\,[t^n] f(t)^{k_2} \qquad (n \geq k_2),$$

thus

$$\begin{aligned}(k_1+k_2)!B_{n,k_1+k_2} &= n!\,[t^n] f(t)^{k_1+k_2} = n!\,[t^n]\Big(f(t)^{k_1} \cdot f(t)^{k_2}\Big)\\ &= n! \sum_{\alpha=0}^{n} [t^\alpha] f(t)^{k_1} \cdot [t^{n-\alpha}] f(t)^{k_2} = n! \sum_{\alpha=0}^{n} \frac{k_1!B_{\alpha,k_1}}{\alpha!} \frac{k_2!B_{n-\alpha,k_2}}{(n-\alpha)!},\end{aligned} \tag{2.5}$$

$$(n \geq k_1 + k_2)$$

since $[t^n]\Big(\phi(t)\psi(t)\Big) = \sum_{\alpha=0}^{n} [t^\alpha]\phi(t) \cdot [t^{n-\alpha}]\psi(t)$ (the Cauchy product of two series). We conclude the proof by noting that the required expression (1.4) follows by rewriting (2.5).

Lastly, we shall prove the closed-form formula (1.5) by making use of (1.4). It suffices to show that the addition formula for $B_{n,k}$ (1.4) may be used to deduce the following:

$$B_{n,2} = \frac{1}{2!}\sum_{\alpha=1}^{n-1}\binom{n}{\alpha} x_{n-\alpha}\, x_{\alpha} \qquad (n \geq 2), \tag{2.6}$$

$$B_{n,3} = \frac{1}{3!}\sum_{\alpha=2}^{n-1}\sum_{\beta=1}^{\alpha-1}\binom{n}{\alpha}\binom{\alpha}{\beta} x_{n-\alpha}\, x_{\alpha-\beta}\, x_{\beta} \qquad (n \geq 3) \tag{2.7}$$

and

$$B_{n,4} = \frac{1}{4!}\sum_{\alpha=3}^{n-1}\sum_{\beta=2}^{\alpha-1}\sum_{\gamma=1}^{\beta-1}\binom{n}{\alpha}\binom{\alpha}{\beta}\binom{\beta}{\gamma} x_{n-\alpha}\, x_{\alpha-\beta}\, x_{\beta-\gamma}\, x_{\gamma} \qquad (n \geq 4). \tag{2.8}$$

By bearing in mind that $B_{n,1} = x_n$ (this is a simple consequence of the definition $B_{n,k}$ in (1.1)) and upon noticing that $x_0 = 0$ (again, see (1.1)), the expression for $B_{n,2}$ given in (2.6) follows by (1.4) with $k_1 = 1$ and $k_2 = 1$. Further, this result for $B_{n,2}$ together with (1.4), where $k_1 = 2$ and $k_2 = 1$, leads to (2.7). It is clear that by repeating this procedure recursively we may obtain $B_{n,4}$, and so on.

3. Further results and concluding remarks

We remark that the explicit closed-form formula for $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ given by (1.5) is particularly useful. Namely, it is hard to work with the formula (1.2) which explicitly defines $B_{n,k}$ due to complicated multiple summations, and, for instance, it is virtually impossible by its use to write down a polynomial for given values of n and k. However, formula (1.5), although also involves multiple summations, makes this possible. In other words, it is now possible to directly evaluate $B_{n,k}$ for given n and k $(n \geq k, k = 2, 3, \ldots)$ by utilizing (1.5) instead of computing it recursively by making use of some recurrence relations (see, for instance, (1.3)). It is noteworthy to mention that the practical evaluation is greatly facilitated by wide availability of various symbolic algebra programs. In order to demonstrate an application of this result, we list several of the polynomials $B_{n,k}$ determined by the formula (1.5), where all the computations were carried out by using Mathematica 6.0 (Wolfram Research)

$$\begin{aligned}
&B_{8,7} = 28x_1^6x_2, \qquad B_{9,7} = 378x_1^5x_2^2 + 84x_1^6x_3,\\
&B_{10,7} = 3150x_1^4x_2^3 + 2520x_1^5x_2x_3 + 210x_1^6x_4,\\
&B_{11,7} = 17325x_1^3x_2^4 + 34650x_1^4x_2^2x_3 + 4620x_1^5x_3^2 + 6930x_1^5x_2x_4 + 462x_1^6x_5,\\
&B_{12,7} = 62370x_1^2x_2^5 + 277200x_1^3x_2^3x_3 + 138600x_1^4x_2x_3^2 + 103950x_1^4x_2^2x_4\\
&\qquad + 27720x_1^5x_3x_4 + 16632x_1^5x_2x_5 + 924x_1^6x_6,\\
&B_{13,7} = 135135x_1x_2^6 + 1351350x_1^2x_2^4x_3 + 1801800x_1^3x_2^2x_3^2 + 200200x_1^4x_3^3\\
&\qquad + 900900x_1^3x_2^3x_4 + 900900x_1^4x_2x_3x_4 + 45045x_1^5x_4^2 + 270270x_1^4x_2^2x_5\\
&\qquad + 72072x_1^5x_3x_5 + 36036x_1^5x_2x_6 + 1716x_1^6x_7.
\end{aligned}$$

It should be noted that our results for $B_{8,7}$ $B_{9,7}$ and $B_{10,7}$ are in full agreement with those recorded in the work (for instance) of Charalambides [4, p. 417].

One further illustration of an application of (1.5) is the following (presumably) new explicit formula

$$S(n,k+1) = \frac{1}{(k+1)!} \underbrace{\sum_{\alpha_1=k}^{n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k} \overbrace{\binom{n}{\alpha_1}\binom{\alpha_1}{\alpha_2}\cdots\binom{\alpha_{k-1}}{\alpha_k}}^{k} \tag{3.1}$$

$$(n \geq k+1, k = 1,2,\ldots)$$

for the Stirling numbers of the second kind $S(n,k)$ defined by means of (see [1, Chapter 5])

$$S(n,k) = \frac{1}{k!}\sum_{\alpha=0}^{k}(-1)^{k-\alpha}\binom{k}{\alpha}\alpha^n, \tag{3.2}$$

which is an immediate consequence of the relationship $S(n,k) = B_{n,k}(1,\ldots,1)$ [1, p. 135, Eq. (3g)]. Moreover, for given k, it is easy to sum the multiple sum (3.1) by repeated use of the familiar result $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$, so that we have:

$$\begin{aligned}
S(n,2) &= \frac{1}{2}\Big(2^n - 2\Big) = 2^{n-1} - 1, \\
S(n,3) &= \frac{1}{6}\Big(3^n - 3\cdot 2^n + 3\Big), \\
S(n,4) &= \frac{1}{24}\Big(4^n - 4\cdot 3^n + 3\cdot 2^{n+1} - 4\Big), \\
S(n,5) &= \frac{1}{120}\Big(5^n - 5\cdot 4^n + 10\cdot 3^n - 10\cdot 2^n + 5\Big), \\
S(n,6) &= \frac{1}{720}\Big(6^n - 6\cdot 5^n + 15\cdot 4^n - 20\cdot 3^n + 15\cdot 2^n - 6\Big),
\end{aligned}$$

and these expressions agree fully with those which are obtained by using the defining relation (3.2).

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